Phys 410
Fall 2014
Lecture #19 Summary
4 November, 2014

The Hamiltonian dynamics formulation is useful for quantum mechanics and for classical statistical mechanics. Here is a way to "derive" quantum mechanics starting from classical mechanics, following the logic of P. A. M. Dirac.

We consider first the Poisson Bracket (PB), which is defined as follows. Consider two dynamical functions of the generalized coordinates and conjugate momenta: $g(\vec{q},\vec{p})$ and $h(\vec{q},\vec{p})$. Examples include angular momentum, linear momentum, mechanical energy, linear kinetic energy, rotational kinetic energy, etc. Define the PB of g,h as $[g,h] \equiv \sum_{i=1}^{n}\left\{\frac{\partial g}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q_i}\right\}$. One can show quite easily that the following statements are true about the PB: $\frac{dg}{dt} = [g,\mathcal{H}] + \frac{\partial g}{\partial t}$, $\dot{q}_j = [q_j,\mathcal{H}]$, $\dot{p}_j = [p_j,\mathcal{H}]$, $[q_j,q_k] = 0$, $[p_j,p_k] = 0$, and most interestingly $[q_j,p_k] = \delta_{kj}$. If the PB of two dynamical quantities vanishes, then the quantities are said to **commute**. If the PB of two dynamical quantities is equal to 1, then the quantities are said to be **canonically conjugate**. Any dynamical quantity that commutes with the Hamiltonian and is not explicitly time dependent is a constant of the motion of the system. Knowing about such quantities can be very useful for understanding the motion of a complex system. Also note from the definition of the PB that $[g,h] = -[h,g]$. Starting with this, Dirac noted that the essential new ingredient of quantum mechanics (QM) is that certain observables $(\hat{u},\hat{v})$ give different answers depending on the order in which the observables operate on a QM system, or in other words $\hat{u}\hat{v} \neq \hat{v}\hat{u}$. To account for this, Dirac re-defined the PB for the quantum case as follows: $i\hbar[u,v]_{QM} \equiv uv - vu$. This leads to the following statements of the "fundamental quantum conditions" for the quantum position and momentum operators: $q_r q_s - q_s q_r = 0$, $p_r p_s - p_s p_r = 0$, and $q_r p_s - p_s q_r = i\hbar\delta_{rs}$. From this statement, one can derive many important results in quantum mechanics, as outlined in Dirac's book *Principles of Quantum Mechanics*.

We then went through the steps of treating the classical Inductor-Capacitor $(L-C)$ parallel circuit as a quantum problem. We chose for the generalized coordinate the flux in the inductor Φ, and employed a mechanical analog to identify the kinetic energy of the system as the energy stored in electric fields in the capacitor $U_C = \frac{1}{2}C\dot{\Phi}^2$, and the energy stored in magnetic fields in the inductor as the potential energy $U_L = \frac{1}{2}\frac{\Phi^2}{L}$. The Lagrangian is $\mathcal{L}(\Phi,\dot{\Phi}) = \frac{1}{2}C\dot{\Phi}^2 - \frac{1}{2}\frac{\Phi^2}{L}$. From this one can find the conjugate momentum as $p = \frac{\partial\mathcal{L}}{\partial\dot{\Phi}} = C\dot{\Phi} = Q$, where Q is the charge on the capacitor plate. Note that the generalized coordinate has the units of flux, while the conjugate momentum has the units of charge! Now we can derive the Hamiltonian and express it in terms of the generalized coordinate and its conjugate momentum as $\mathcal{H}(\Phi,\mathrm{Q}) = \frac{Q^2}{2C} + \frac{1}{2}\frac{\Phi^2}{L}$. Dirac's prescription is to observe that the coordinate and its conjugate momentum have a Poisson bracket of unity, hence the corresponding QM operators have this relationship: $\widehat{\Phi}\widehat{\mathrm{Q}} - \widehat{\mathrm{Q}}\widehat{\Phi} = i\hbar[\Phi,\mathrm{Q}]$, with $[\Phi,\mathrm{Q}] = 1$. The Hamiltonian can be re-written in terms of the creation and annihilation operators as $\widehat{\mathcal{H}} = \hbar\omega\left(\hat{a}^{+}\hat{a} + \frac{1}{2}\right)$, which is the Hamiltonian of a harmonic oscillator with $\omega = 1/\sqrt{LC}$.

We next considered the most general motion of systems of particles. We specifically consider rigid bodies, defined as multi-particle objects in which the distance between any two particles never changes as the object moves. As discussed before, this puts a huge constraint on the system, changing it from a 3N degree of freedom object to a 6 degree of freedom system. We reviewed the center of mass, center of mass momentum, and Newton's second law for the CM. We then considered the angular momentum of a rigid body and found that it decomposes cleanly into the angular momentum of the center of mass (relative to some origin), and the angular momentum relative to the CM. For a rigid body, the only motion it can have relative to the CM is rotation. A similar observation was made about the kinetic energy of the rigid body.